1.1 Proofs

1. Suppose $\sqrt{6} \in \mathbb{Q}$ (= the rationals). Then $\exists\ m$ and n in $\mathbb{N}$ such that $\sqrt{6} = \frac{m}{n}$ and $\frac{m}{n}$ is in lowest terms. Hence, $6n^2 = m^2$. Since 2 divides $6n^2$, 2 divides m^2. Since 2 is a prime, 2 divides m. Thus, $\exists\, k \in \mathbb{N}$ such that $m = 2k$, and so $6n^2 = 4k^2$ or $3n^2 = 2k^2$. As above, 2 divides $3n^2$. Since 2 does not divide 3 and 2 is a prime, 2 divides n. Therefore, $\frac{m}{n}$ is not in lowest terms, a contradiction; so $\sqrt{6} \notin \mathbb{Q}$.

2. Let p be a prime and suppose $\sqrt{p} \in \mathbb{Q}$. Then $\exists\ m$ and n in $\mathbb{N}$ such that $\sqrt{p} = \frac{m}{n}$ and $\frac{m}{n}$ is in lowest terms. Hence, $pn^2 = m^2$. Since p is a prime, as in Exercise 1, p divides m. Thus $\exists\ k \in \mathbb{N}$ such that $m = pk$, and so $pn^2 = p^2k^2$ or $n^2 = pk^2$. As above, p divides n and so m/n is not in lowest terms, a contradiction. Therefore, $\sqrt{p} \notin \mathbb{Q}$.

3. If $a = b = 0$, then $a^2 + b^2 = 0 + 0 = 0$. If $a \neq 0$, then $a^2 + b^2 \geq a^2 > 0$. The case $b \neq 0$ is satisfied by symmetry.

4. If $a = 0$, then $ab = 0(b) = 0$ and symmetry takes care of $b = 0$. If $a \neq 0$ and $b \neq 0$ (it may be worth pointing out that this is the negation of $a = 0$ or $b = 0$), then $\frac{1}{a}$ and $\frac{1}{b}$ are real numbers and $(ab)\left(\frac{1}{a}\frac{1}{b}\right) = 1$. Thus, $ab \neq 0$.

5. If $a < 0$ and $-a \leq 0$, then $0 = -a + a < 0$, a contradiction.

6. If $a > 0$ and $\frac{1}{a} \leq 0$, then $1 = a \cdot \frac{1}{a} \leq a \cdot 0 = 0$, a contradiction.

7. Suppose $a^2 = a$. If $a \neq 0$, then $\frac{1}{a} \in \mathbb{R}$ and $a = \frac{1}{a} \cdot a^2 = \frac{1}{a} \cdot a = 1$. (It may not be evident to some students that this completes the argument.) Alternatively, $a^2 = a \Rightarrow a^2 - a = 0 \Rightarrow a(a-1) = 0$. By Exercise 4, either $a = 0$ or $a - 1 = 0$, and so either $a = 0$ or $a = 1$.

8. In the notation of the hint, $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \leq \frac{\sum_{i=1}^{n} x_k}{n} = \frac{nx_k}{n} = x_k$. (We are of course assuming that any nonempty finite set of real numbers has a maximum. See Exercise 2.2.8.)

1.2 Sets

1. For the second equality in part 6, let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. So either $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. Therefore, $x \in (A \cap B) \cup (A \cap C)$ and so $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. For the other containment, let $x \in (A \cap B) \cup (A \cap C)$. So either $x \in A \cap B$ or $x \in A \cap C$. If $x \in A \cap B$, then $x \in A$ and $x \in B$ implying that $x \in A$ and $x \in B \cup C$ since $B \subset B \cup C$ by part 3. If $x \in A \cap C$, then $x \in A$ and $x \in C$ implying that $x \in A$ and $x \in C \cup B = B \cup C$ by parts 3 and 4. In either case $x \in A \cap (B \cup C)$.
   For the second equality in part 7, first note that $A \cap B \subset B$. So if $A = A \cap B$, then $A \subset B$. Next suppose $A \subset B$. Since $A \cap B \subset A$, we have to show that $A \subset A \cap B$. If $x \in A$, since $A \subset B$, $x \in B$. Hence, $x \in A \cap B$.

2. To show the second equality in part 2, let $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. So $x \in A$ and $x \notin B$ and $x \notin C$. Now $x \in A$ and $x \notin B$ implies $x \in A \setminus B$ while $x \in A$ and $x \notin C$ implies $x \in A \setminus C$. Hence $x \in (A \setminus B) \cap (A \setminus C)$. For the other containment, reverse the above steps.

3. Let $x \in X \setminus \bigcap_{\alpha \in I} A_\alpha$. Then $x \in X$ and $x \notin \bigcap_{\alpha \in I} A_\alpha$. So $\exists\ \alpha_0 \in I$ such that $x \notin A_{\alpha_0}$. Then $x \in X \setminus A_{\alpha_0}$ and so $x \in \bigcup_{\alpha \in I}(X \setminus A_\alpha)$. To show the other containment, let $x \in \bigcup_{\alpha \in I}(X \setminus A_\alpha)$. Then $\exists\ \alpha_0 \in I$ with $x \in X \setminus A_{\alpha_0}$. So $x \in X$ and $x \notin A_{\alpha_0}$. So $x \in X$ and $x \notin \bigcap_{\alpha \in I} A_\alpha$. Therefore, $x \in X \setminus \bigcap_{\alpha \in I} A_\alpha$.

4. If $x \in (A \cup B) \setminus (A \cap B)$, then $x \in A \cup B$ and $x \notin A \cap B$. Thus, either $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$, and so either $x \in A \setminus B$ or $x \in B \setminus A$. Therefore, $x \in (A \setminus B) \cup (B \setminus A)$. Reverse these steps for the other containment.

5. If $x \in B \setminus (B \setminus A)$, then $x \in B$ and $x \notin B \setminus A$. If $x \notin A$, then $x \in B \setminus A$. Therefore $x \in A$. Thus, $B \setminus (B \setminus A) \subset A$. If $x \in A$, then $x \notin B \setminus A$. Since $A \subset B$, $x \in B$. So, $x \in B \setminus (B \setminus A)$.

6. Since $A \subset B$ and $B \setminus A \subset B$, $A \cup (B \setminus A) \subset B$. To show $B \subset A \cup (B \setminus A)$, let $x \in B$. Either $x \in A$ or $x \notin A$. So either $x \in A$ or $x \in B \setminus A$.

7. Let $A = \{1\}, B = \{1,2\}, C = \{1,3\}$.

8. By drawing these sets it should be clear that $\bigcup_{n\in\mathbb{N}} A_n = (1,\infty)$. $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$ for if $x \in \bigcap_{n\in\mathbb{N}} A_n$, then $x > n\ \forall\ n \in \mathbb{N}$.

9. A picture easily indicates that $\bigcup_{n\in\mathbb{N}} A_n = [0,1]$. $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$ for if $x > 0$, by the Archimedean Principle, $\exists\ n_0$ in $\mathbb{N}$ with $\frac{1}{n_0} < x$ and hence $x \notin A_{n_0}$.

10. For the first equality

$$\begin{aligned} x &\in X \cap (\bigcup_{\alpha\in I} A_\alpha) \Leftrightarrow x \in X \text{ and } x \in \bigcup_{\alpha\in I} A_\alpha \\ &\Leftrightarrow x \in X \text{ and } x \in A_{\alpha_0} \text{ for some } \alpha_0 \in I \\ &\Leftrightarrow x \in X \cap A_{\alpha_0} \text{ for some } \alpha_0 \in I \\ &\Leftrightarrow x \in \bigcup_{\alpha\in I}(X \cap A_\alpha). \end{aligned}$$

To show $X \cup (\bigcap_{\alpha\in I} A_\alpha) \subset \bigcap_{\alpha\in I}(X \cup A_\alpha)$, let $x \in X \cup (\bigcap_{\alpha\in I} A_\alpha)$. So either $x \in X$ or $x \in \bigcap_{\alpha\in I} A_\alpha$. If $x \in X$, then $x \in X \cup A_\alpha\ \ \forall\alpha \in I$. If $x \in \bigcap_{\alpha\in I} A_\alpha$, then $x \in A_\alpha\ \forall\alpha \in I$ and so $x \in X \cup A_\alpha\ \forall\alpha \in I$. In either case, $x \in \bigcap_{\alpha\in I}(X \cup A_\alpha)$.

For the reverse containment, let $x \in \bigcap_{\alpha\in I}(X \cup A_\alpha)$. So $x \in X \cup A_\alpha\ \forall\alpha \in I$. We may assume $x \notin X$ (for otherwise we have the desired containment). Then $x \in A_\alpha\ \forall\alpha \in I$ and so
$x \in \bigcap_{\alpha\in I} A_\alpha \subset X \cup (\bigcap_{\alpha\in I} A_\alpha)$.

11. Let $(x,y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. So either $y \in B$ or $y \in C$. If $x \in A$ and $y \in B$, then $(x,y) \in A \times B$ while if $x \in A$ and $y \in C$, then $(x,y) \in A \times C$. Therefore, $(x,y) \in (A \times B) \cup (A \times C)$.

For the other containment, let $(x,y) \in (A \times B) \cup (A \times C)$. If $(x,y) \in A \times B$, then $x \in A$ and $y \in B \subset B \cup C$. Similarly, if $(x,y) \in A \times C$, then $x \in A$ and $y \in C \subset B \cup C$. Thus $(x,y) \in A \times (B \cup C)$.

1.3 Functions

1. If $y \in f(A \cup B)$, then $y = f(x)$ for some $x \in A \cup B$. If $x \in A$, then $f(x) \in f(A)$ while if $x \in B$, then $f(x) \in f(B)$. Thus, $y = f(x) \in f(A) \cup f(B)$ and so $f(A \cup B) \subset f(A) \cup f(B)$.